Risk Measures for Solvency Regulation and Asset Liability Management

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Motivation

- A key task of insurance companies consists in managing and balancing **risk and return**
- Important quantities are, for example, the
  - **Market-Consistent Embedded Value** (MCEV)

    as a quantitative measure of the value of the current business

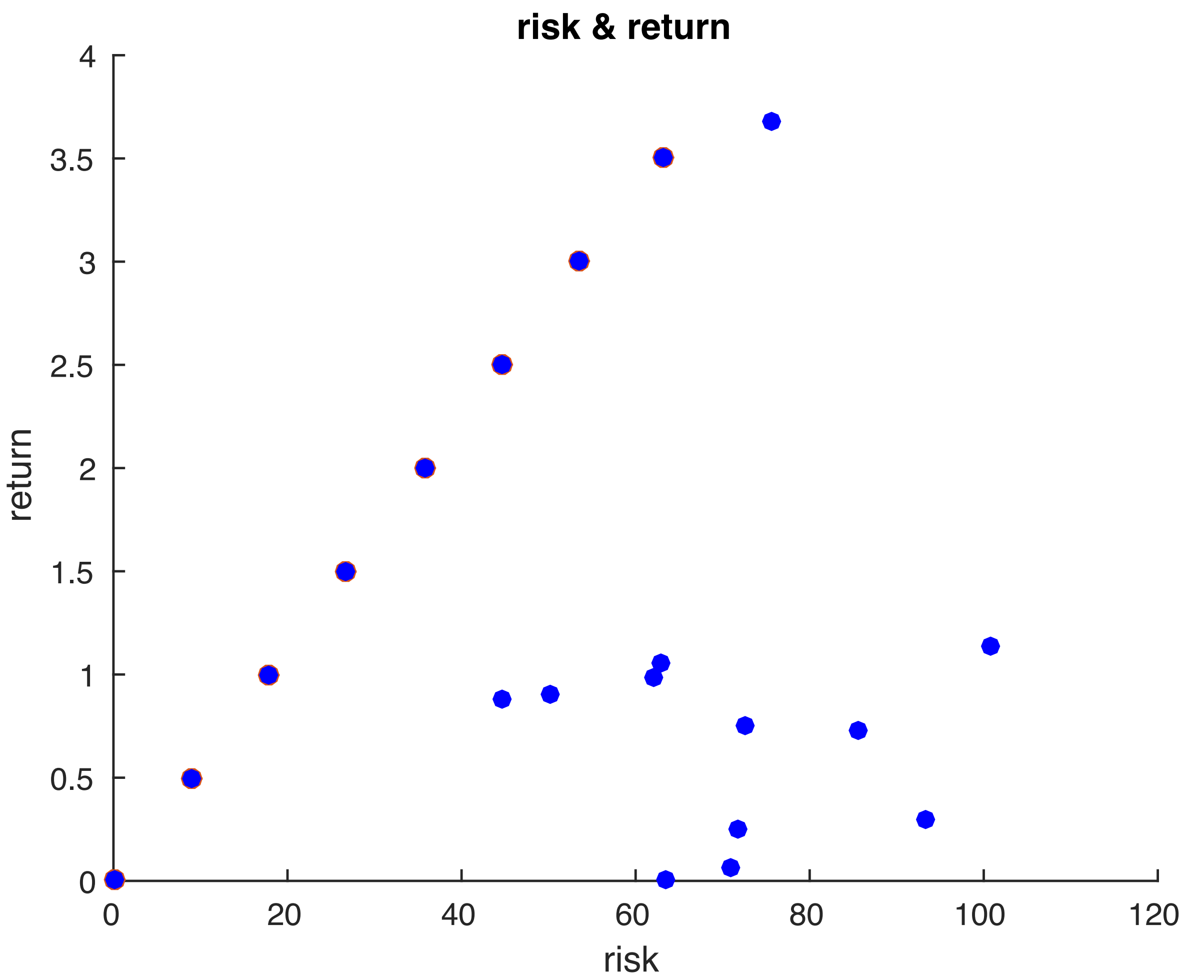
    and
  - **Value at Risk** (V@R) or **Average Value at Risk** (AV@R)

    as quantitative measures of the downside risk.

Motivation (2)

- The optimization of risk and return requires adequate strategies for the management of the assets and the liabilities of the balance sheet.
- A comprehensive analysis is quite sophisticated:
  - Quantities like MCEV (VIF, FS, ReC), SCR,... need to be computed.
  - The impact of dynamic management rules must be characterized.
  - This requires adequate ALM-models.

Efficient Frontier



Outline

(i) Short review of Solvency II – Pillar I:
Quantitative Requirements

(ii) Alternative risk measures:
AV@R, UBSR

(iii) Statistical risk measurement:
Risk regression and robustness

(iv) Future research:
Systemic risk and group risk

Solvency II

Solvency II

- Main aim: more appropriate and better risk measurement
- Methodology:
  - Standard formula or internal model
  - Market or market-consistent values instead of accounting measures
  - Three pillar approach

Three Pillars

- **I. Quantitative Requirements**
- **II. Supervisory Review**
- **III. Disclosure**

Three Pillars (2)

- **I. Quantitative Requirements**
  - Market-consistent valuation of assets and liabilities
  - Computation of the solvency capital requirement

  on the basis of the standard formula or an internal model
- II. Supervisory Review
- III. Disclosure

The Solvency Balance Sheet

- The role of capital
  - Buffer for potential losses
  - that protects policy holders (and other counterparties)
- Solvency balance sheet
  - Market-consistent valuation of all assets and liabilities
- SCR = Solvency Capital Requirement
  - Key goal: Limit one-year probability of ruin to at most 0.5%.

The Solvency Balance Sheet (2)

Assets

- Market value of all assets

Liabilities

- Economic capital
  - i.e. SCR + Free Surplus
- Non-hedgeable liabilities
  - Best Estimate
  - Risk Margin
- Hedgeable liabilities
  - Market-Consistent Value

SCR

The SCR corresponds to the economic capital a (re)insurance undertaking needs to hold in order to limit the probability of ruin to 0.5%, i.e. ruin would occur once every 200 years. . .

The SCR is calculated using Value-at-Risk techniques, either in accordance with the standard formula, or using an internal model: all potential losses, including adverse revaluation of assets and liabilities, over the next 12 months are to be assessed. The SCR reflects the true risk profile of the undertaking, taking account of all quantifiable risks, as well as the net impact of risk mitigation techniques.

(Proposal for a Directive of the European Parliament and of the Council on the taking-up and pursuit of the business of Insurance and Reinsurance - Solvency II, COMMISSION OF THE EUROPEAN COMMUNITIES, Brussels, 10.7.2007)

SCR in a Simplified Internal Model

- **Time**: $t = 0, 1$
- **Value of assets**: A_t, $t = 0, 1$
- **Value of liabilities**: L_t, $t = 0, 1$
- **Capital** (NAV): $E_t = A_t - L_t$, $t = 0, 1$
- **Level**: $\alpha = 0.05$

$$P(E_1 \le 0) \quad \le \quad \alpha$$

$$\Leftrightarrow \; V@R_\alpha(E_1) \le 0 \; \Leftrightarrow \; V@R_\alpha(E_1 - E_0) \le E_0 \quad \Leftrightarrow \quad V@R_\alpha(\Delta A_1 - \Delta L_1) \le E_0,$$

with $\Delta A_1 = A_1 - A_0$, $\Delta L_1 = L_1 - L_0$.

Cash flows (premia, taxes, etc.) are assumed to be implicitly included.
Simplified NAV instead of MCEV computation, thus VIF neglected.

SCR in a Simplified Internal Model (2)

As a consequence the **SCR** is defined as follows:

$$\mathbf{SCR} = \mathbf{V@R}_\alpha(\mathbf{\Delta A_1} - \mathbf{\Delta L_1}) = \mathbf{V@R}_\alpha(\mathbf{\Delta NAV})$$

The **solvency condition** can be reformulated:

$$\mathbf{SCR} \leq \mathbf{E_0}.$$

Alternative Risk Measures

Monetary Risk Measures

- Model for one time period as in Solvency II: $t = 0, 1$
- $\mathcal{X}$ is space of positions at time 1 modeled by random variables (P&L)

Risk measures

$\rho : \mathcal{X} \to \mathbb{R}$

- Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- Cash invariance: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.

A risk measure is statistics that summarizes certain properties of random future balance sheets.

Risk measures like V@R focus on the downside risk.

Capital requirements

- A position $X \in \mathcal{X}$ is acceptable, if $\rho(X) \leq 0$.
  The collection $\mathcal{A}$ of all acceptable positions is the *acceptance set*.

- ρ is a capital requirement, i.e.

$$\rho(X) = \inf \{m \in \mathbb{R} : \; X + m \in \mathcal{A}\} .$$

Example

$$\mathrm{V@R}_\lambda(X) = \inf\{m \in \mathbb{R} : \; P[m + X < 0] \leq \lambda\}$$

"Smallest monetary amount that needs to be added to a position such that the probability of a loss becomes smaller than λ."

Diversification

Semiconvexity:

$$\rho(\alpha X + (1-\alpha)Y) \leq \max(\rho(X), \rho(Y)) \qquad (\alpha \in [0,1]).$$

$\Longrightarrow$

Convexity (Föllmer & Schied, 2002):

$$\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y) \qquad (\alpha \in [0,1]).$$

Average Value at Risk

$$\mathrm{AV@R}_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda \mathrm{V@R}_\gamma(X)\,d\gamma$$

Properties

- coherent (i.e. convex and positively homogeneous)
- sensitive to large losses
- basis of Swiss Solvency test
- common alternative to V@R in practice
- distribution-based and continuous from above
- building block of large class of risk measures

Utility-based Shortfall Risk (UBSR)

$\ell : \mathbb{R} \to \mathbb{R}$ convex loss function, z interior point of the range of ℓ.

The acceptance set is defined as

$$\mathcal{A} = \{X \in L^\infty : \ E_P [\ell(-X)] \leq z\}$$

$\mathcal{A}$ induces a convex risk measure ρ:

$$\rho(X) = \inf\{m \in \mathbb{R} : \ X + m \in \mathcal{A}\}$$

Simple formula

Shortfall risk $\rho(X)$ is given by the unique root s_* of the function

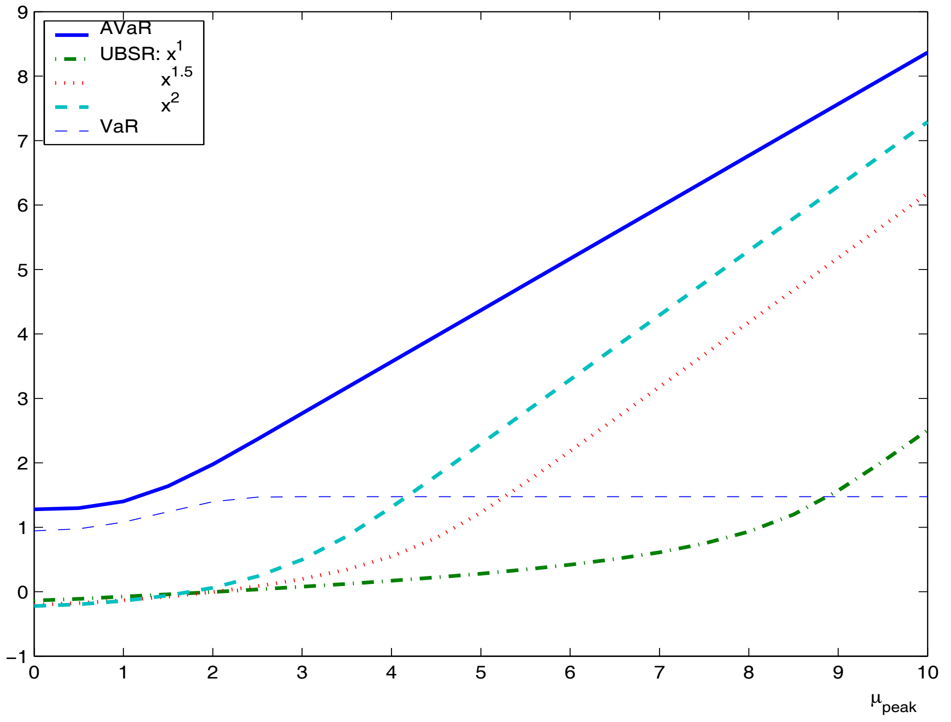
$$f(s) := E[\ell(-X - s)] - z.$$

Utility-based Shortfall Risk (UBSR)

Properties

- convex
- sensitive to large losses
- distribution-based and continuous from above
- easy to estimate and implement (see below)
- elicitable (see below)

Sensitivity to the Downside Risk



$\mathrm{V@R}_{0.05}$, $\mathrm{AV@R}_{0.05}$ and UBSR with $p \in \{1, \frac{3}{2}, 2\}$ and $z = 0.3$ of a mixture of a Student t (weight 0.96) and a Gaussian with mean μ (weight 0.04) as a function of μ.

Value at Risk in the Media

"David Einhorn, who founded Greenlight Capital, a prominent hedge fund, wrote not long ago that VaR was

> '*relatively useless as a risk-management tool and potentially catastrophic when its use creates a false sense of security among senior managers and watchdogs. This is like an air bag that works all the time, except when you have a car accident.*' "

"Nicholas Taleb, the best-selling author of 'The Black Swan,' has crusaded against VaR for more than a decade. He calls it, flatly, '*a fraud.*' "

("Risk Mismanagement", New York Times, 2. Januar 2009)

Application to SCR

SCR in a Simplified Internal Model

- **Time**: $t = 0, 1$
- **Value of assets**: A_t, $t = 0, 1$
- **Value of liabilities**: L_t, $t = 0, 1$
- **Capital** (NAV): $E_t = A_t - L_t$, $t = 0, 1$

$$\begin{aligned} & P(E_1 \leq 0) \leq \alpha \\ \Leftrightarrow \quad & E_1 \in \mathcal{A}_{\mathrm{V@R}_\alpha} \\ \Leftrightarrow \quad & SCR := \mathrm{V@R}_\alpha(\Delta A_1 - \Delta L_1) \leq E_0, \end{aligned}$$

with $\Delta A_1 = A_1 - A_0$, $\Delta L_1 = L_1 - L_0$.

Generalized SCR

- **Time**: $t = 0, 1$
- **Value of assets**: A_t, $t = 0, 1$
- **Value of liabilities**: L_t, $t = 0, 1$
- **Capital** (NAV): $E_t = A_t - L_t$, $t = 0, 1$

$$\begin{aligned} & \rho(E_1) \leq 0 \\ \Leftrightarrow \quad & E_1 \in \mathcal{A}_\rho \\ \Leftrightarrow \quad & SCR := \rho(\Delta A_1 - \Delta L_1) \leq E_0, \end{aligned}$$

with $\Delta A_1 = A_1 - A_0$, $\Delta L_1 = L_1 - L_0$.

Generalized SCR (2)

- **Requirement**: E_1 acceptable with respect to risk measure ρ
- **Solvency capital requirement**

  is equivalent to

$$\rho(\mathbf{\Delta NAV}) = \rho(\mathbf{\Delta A_1} - \mathbf{\Delta L_1}) \leq \mathbf{E_0}$$

(Proof: as previous computation)

Example: SCR and AV@R

Using $\mathbf{AV@R}$ (which is used in the SST) generates the SCR:

$$\mathbf{AV@R}_\lambda(\mathbf{\Delta NAV}) = \mathbf{AV@R}_\lambda(\mathbf{\Delta A_1} - \mathbf{\Delta L_1})$$

In this case, the solvency requirement does not focus on the probability of insolvency, but on the average loss in the case of insolvency.

- From a macroeconomic point of view, this seems to be a reasonable approach.
- Alternatively, **UBSR** could be used instead of V@R or AV@R.

Statistics of Risk: Backtesting and Robustness

Motivation

Risk measures should possess adequate statistical properties for their estimation and a comparison to data:

- **Elicitability**
  - Good properties in the context of backtesting? (Gneiting, 2011)
  - Generalized quantile regression methods (Koenker, 2005)
- **Robustness**
  - Robust computation in case of slightly incorrect models

Elicitability

A scoring function $S : \mathbb{R}^2 \to [0, \infty)$ satisfies:

- $S(x, y) \geq 0$, and $S(x, y) = 0 \Leftrightarrow x = y$
- $x \mapsto S(x, y)$ increasing for $x > y$ and decreasing for $x < y$
- $x \mapsto S(x, y)$ continuous

Definition 1 *A statistical functional $T : \mathcal{M} \to \mathbb{R}$ is* **elicitable** *on $\mathcal{M}$, if there exists a scoring function S such that for each $\mu \in \mathcal{M}$:*

$$\int S(x, y)\mu(dy) < \infty,$$

$$\mathbf{T}(\mu) = \mathrm{argmin}_{\mathbf{x} \in \mathbb{R}} \int \mathbf{S}(\mathbf{x}, \mathbf{y})\mu(\mathbf{dy})$$

Characterization

- An application of results of W. (2006) implies a complete characterization of all **elicitable risk measures** under weak technical conditions on the scoring function S.
- The details are worked out in Bellini & Bignozzi (2015).
- Delbaen, Bellini, Bignozzi & Ziegel (2015) investigate the special case of convex risk measures which does not require additional topological assumptions.

Characterization (2)

Theorem 1 *Let ρ be a risk measure that is elicitable for a regular scoring function.*

Then the following statements hold:

(i) ρ is convex, if and only if ρ is UBSR.

(ii) ρ is coherent, if and only if ρ is an expectile.

Conversely, UBSR is always elicitable for a regular scoring function.

Remark

AV@R is not elicitable. (W., 2006; Gneiting, 2011)

Expectile

Special case of coherent UBSR with piecewise linear loss function

$$\ell(x) = z + \alpha x^{+} - \beta x^{-},$$

$x \in \mathbb{R}$, $\alpha \geq \beta > 0$ with level $z \in \mathbb{R}$.

Acceptability

A position $X \in L^{\infty}$ is acceptable, if and only if

$$\beta E(X^{+}) - \alpha E(X^{-}) \geq 0,$$

i.e. a difference between weighted expected gains and weighted expected losses is larger than 0.

Remark

AV@R and expectiles are not surplus invariant.

Robustness

Robustness

- Convex risk measures have recently been criticized, since they are not Hampel-robust, see e.g. Cont, Deguest & Scandolo (2010) and Kou, Peng & Heyde (2013).
- This notion of robustness is, however, related to the weak topology and formalized in terms of a metrization like the Lévy or Prohorov metric which are tail-insensitive.
- By Hampel's theorem, Hampel-robustness implies continuity with respect to the weak topology and, thus, insensitivity to the tails.

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Question:

Should we measure the downside risk of financial positions with tail-insensitive, but Hampel-robust functionals?

Robustness (2)

- Robustness depends on the metric that is used on the space of probability measures.
- Risk measures are not either robust or not robust, but more or less robust.
- $\exists$ tradeoff between tail sensitivity and robustness.
- These issues were studied by Krätschmer, Schied & Zähle (2014).
- Here: application in the context of **elicitable** risk measures.

Distribution-based Risk Measures

A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is distribution-based, if

$$\rho(X) = \rho(Y),$$

whenever $\mathcal{L}(X) = \mathcal{L}(Y)$.

Risk measures on the level of distributions

In this case, the risk measure defines a map

$$\mathcal{R}_\rho(\mu) := \rho(X) \text{ with } \mathcal{L}(X) = \mu$$

on Borel probability measures.

Qualitative Robustness

- Let $\Omega = \mathbb{R}^{\mathbb{N}}$, $X_i(\omega) = \omega(i)$, and $\mathcal{F} = \sigma(X_i : i \in \mathbb{N})$.
- For any Borel probability measure μ on $\mathbb{R}$ denote

$$P_\mu := \mu^{\otimes \mathbb{N}}.$$

- $\mathcal{N} \subseteq \mathcal{M}_1$ with metric d_A
- d_B metric on $\mathcal{M}_1$

$\mathcal{R}_\rho$ robust on $\mathcal{N}$ with respect to d_A, d_B, if

$$\forall \mu \in \mathcal{N}, \varepsilon > 0 \; \exists \delta > 0, n_0 \in \mathbb{N} \; \forall n \geq n_0 :$$

$$\nu \in \mathcal{N}, d_A(\mu, \nu) \leq \delta \Rightarrow d_B(P_\mu \circ \hat{\rho}_n^{-1}, P_\nu \circ \hat{\rho}_n^{-1}) \leq \varepsilon$$

where $\hat{\rho}_n := \mathcal{R}_\rho(\frac{1}{n}\sum_{k=1}^n \delta_{X_k})$

ψ-Robustness

A set $\mathcal{N} \subset \mathcal{M}_1$ is called uniformly ψ-integrating, if

$$\lim_{M \to \infty} \sup_{\nu \in \mathcal{N}} \int_{\{\psi \geq M\}} \psi d\nu = 0.$$

Definition 2

A risk functional $\mathcal{R}_\rho$ is called ψ-robust on $\mathcal{M} \subset \mathcal{M}_1$,

if it is robust with respect to d_ψ and d_{Proh} on every uniformly ψ-integrating set $\mathcal{N} \subset \mathcal{M}$.

ψ-Robustness (2)

The Prohorov-ψ-metric is

$$d_\psi(\mu,\nu) = d_{\text{Proh}}(\mu,\nu) + \left| \int \psi d\mu - \int \psi d\nu \right|, \quad \mu,\nu \in \mathcal{M}_1^\psi,$$

where the Prohorov-metric is defined by

$$d_{\text{Proh}}(\mu,\nu) = \inf\{\varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ for all } A \in \mathcal{B}(\mathbb{R})\}$$

with $A^\varepsilon = \{x \in \mathbb{R} : \inf_{a\in A} |x-a| \leq \varepsilon\}$, $\varepsilon > 0$.

While the weak topology is induced by the Prohorov-metric and is insensitive with respect to extreme events, the ψ-weak topology is sensitive with respect to such events.

ψ-Robustness (3)

Theorem 2 *Let $\rho : L^\infty \to \mathbb{R}$ be a distribution-based risk measure and $\bar{\rho} : L^1 \to \mathbb{R} \cup \{+\infty\}$ its unique extension. Suppose that Ψ is a finite Young-function that satisfies the Δ_2-condition. Define $\psi(x) = 1 + \Psi(|x|)$, $x \in \mathbb{R}$.*

Then the following conditions are equivalent:

- $\bar{\rho}$ is *finite on H^Ψ*.
- *The mapping $\mathcal{R}_\rho : \mathcal{M}_{1,c} \to \mathbb{R}$ is continuous with respect to the ψ-weak topology.*
- $\mathcal{R}_\rho : \mathcal{M}_{1,c} \to \mathbb{R}$ is *ψ-robust*.

Index of Qualitative Robustness

L^p-spaces are particularly important for applications. Let $\Psi_p(x) = |x|^p/p$ with $0 < p < \infty$, $x \in \mathbb{R}$, then the Δ_2-condition is always satisfied.

Definition 3 *Let $\rho : L^\infty \to \mathbb{R}$ be a distribution-based risk measure. The associated* index of qualitative robustness *is defined as*

$$\mathrm{iqr}(\rho) \;=\; \frac{1}{\inf\{p \in (0,\infty) : \mathcal{R}_\rho \text{ is } \psi_p - \text{robust on } \mathcal{M}_{1,c}\}} \;\in\; [0,\infty]$$

Convex risk measures have an index of qualitative robustness of at most 1. Theorem 2 implies the following formula:

$$\mathrm{iqr}(\rho) \;=\; \frac{1}{\inf\{p \in [1,\infty) : \bar{\rho} \text{ is finite on } L^p\}} \;\in\; [0,1] \qquad (1)$$

Elicitability and Robustness

Convex elicitable risk measures coincide with the class of utility-based shortfall risk measures:

$$\rho(X) = \inf \{m \in \mathbb{R} : E(\ell(-X - m)) \leq z\}$$

with convex, increasing, non-constant function $\ell : \mathbb{R} \to \mathbb{R}$ and z in the interior of the range of ℓ.

- Set $\Psi(x) = \ell(x) - \ell(0)$, $x \geq 0$, $\psi(x) = \Psi(|x|) + 1$, $x \in \mathbb{R}$. Assume that the function Ψ satisfies the Δ_2-condition.
- Elementary bounds show that $\mathcal{R}_\rho$ is continuous with respect to the ψ-weak topology and thus ψ-robust.

Elicitability and Robustness (2)

Expectiles

Special case of UBSR with piecewise linear loss function
$\ell(x) = z + \alpha x^+ - \beta x^-$, $x \in \mathbb{R}$, $\alpha \geq \beta > 0$ with level $z \in \mathbb{R}$.

- Finite distribution-based risk measures on L^1.
- Index of qualitative robustness is 1.
- Expectiles are as robust as Average Value at Risk.

Elicitability and Robustness (3)

Monomials

For $\ell(x) = x^p \mathbf{1}_{\{x \geq 0\}}$ with Level $z > 0$ the associated UBSR has the following properties:

- ρ is finite on L^p with an index of qualitative robustness of $1/p$.
- The corresponding functional $\mathcal{R}_\rho$ is ψ_p-robust.

Entropic risk measure

- The entropic risk measure is never finite on L^p, $1 \leq p < \infty$.
- Equation (1) shows that its index of qualitative robustness is 0.

Future Research

Systemic Risk and Group Risk

- Key objectives of the regulation of financial systems and the management of insurance groups are upper bounds on overall risk in the system or group.
- Capital levels for individual entities cannot be chosen independently of each other, but jointly influence overall risk.
- A similar effect occurs when a company has subsidiaries in countries with different currencies.
- These issues can be captured by set-valued risk measures.
- Systemic risk measures are constructed in Feinstein, Rudloff & W. (2015); group risk is considered by Michael Schmutz in the next talk.

Conclusion

Conclusion

- V@R has significant deficiencies. These deficiencies are inherited by the SCR-regulation under Solvency II.
- AV@R and expectiles are reasonable additional functionals that characterize the downside risk and can easily be implemented.

Recommendation

The analysis of the downside risk should involve a joint analysis of V@R, AV@R and expectiles.

Future Research

Systems of entities require set-valued risk measures for a proper characterization of their risk.

Selected References

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